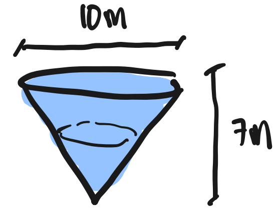
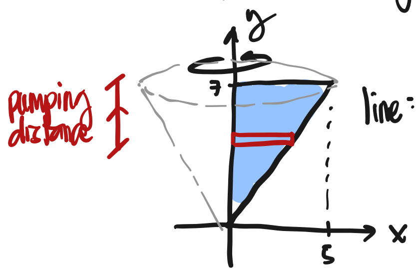
## Example ①:




Water Tank: Inverted Right Circular Cone that is 10m across the top and 7m high.

Determine the work done by pumping the water to the top of the tank, assuming the tank is full.

**Method 1.** Model the tank as a solid of revolution with region: bounded by the y-axis, $y = 7$ and the line passing through the origin and the point $(5,7)$;




line: $y = \frac{7}{5}x$

thickness: $dy$, bounds: $y \in [0,7]$

volume of slice: $dV = \pi r^2\, dy$ with $r = x$ and $y = \frac{7}{5}x$;

$x = \frac{5}{7}y$; $dV = \pi\left(\frac{5}{7}y\right)^2 dy$

pumping distance: $7 - y$

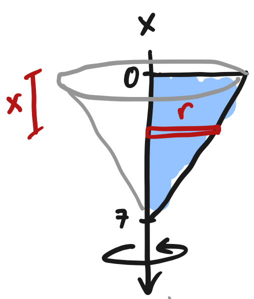
density of water: $1000\ kg/m^3$, acc. due to gravity: $9.8\ m/s^2$

$$V = \int_0^7 9800 \left(\begin{matrix}\text{displacement}\\ \text{of slice}\end{matrix}\right)\left(\begin{matrix}\text{volume}\\ \text{of slice}\end{matrix}\right) = \int_0^7 9800(7-y)(\pi)\left(\frac{5}{7}y\right)^2 dy$$

$$= 9800\pi\left(\frac{25}{49}\right)\int_0^7 7y^2 - y^3\, dy = 9800\pi\left(\frac{25}{49}\right)\left[\frac{7}{3}y^3 - \frac{1}{4}y^4\right]_0^7$$

$$= 9800\pi\left(\frac{25}{49}\right)\left[\frac{7^4}{3} - \frac{7^4}{4}\right] = \frac{3\,001\,250}{3}\pi \approx \underline{3\ 142\ 902\ J}$$

**Method 2.** Model the tank using similar triangles.

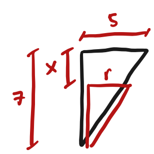


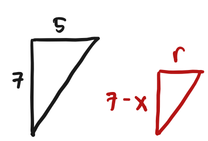

Note: x-axis increases from top to bottom.

thickness: $dx$, bounds: $x \in [0,7]$

volume of disk: $dV = \pi r^2\, dx$;

To find $r$ in terms of $x$, consider these 2 similar triangles:




$\frac{r}{7-x} = \frac{5}{7}$; $r = \frac{5}{7}(7-x)$;

$dV = \pi r^2 dx = \pi\left(\frac{5}{7}(7-x)\right)^2 dx = \frac{25\pi}{49}(7-x)^2 dx$

$= \frac{25\pi}{49}(49 - 14x + x^2)\, dx$

pumping distance: $x$

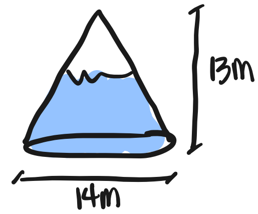
$$V = 9800\int \left(\begin{matrix}\text{pumping}\\ \text{dist. of slice}\end{matrix}\right) dV = 9800\int_0^7 (x)\left(\frac{25\pi}{49}\right)(49 - 14x + x^2)\, dx$$

$$= 9800\pi\left(\frac{25}{49}\right)\int_0^7 49x - 14x^2 + x^3\, dx = \frac{9800\pi(25)}{49}\left[\frac{49}{2}x^2 - \frac{14}{3}x^3 + \frac{1}{4}x^4\right]_0^7$$

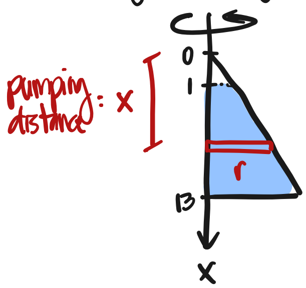
$$= 9800\pi\left(\frac{25}{49}\right)\left[\frac{49}{2}(7^2) - \frac{14}{3}(7^3) + \frac{1}{4}(7^4) - 0\right] = \frac{3\,001\,250}{3}\pi \approx \underline{3\ 142\ 902\ J}$$

Example ②

Consider a water tank with the shape of a circular cone with base diameter 14 m and height 13 m, oriented. Determine the work $W$ done by pumping the water (assuming the water has a depth of 9 m) to the top of the cone.

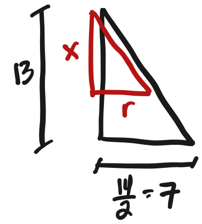



Method 1. Use the following model using similar triangles.




thickness: $dx$, bounds: $x \in [1, 13]$

volume of slice: $dV = \pi r^2 \, dx$;




$\frac{r}{x} = \frac{7}{13}$; $r = \frac{7}{13}x$;

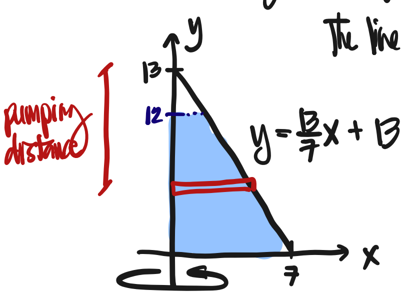
Then, $dV = \pi r^2 \, dx = \pi\left(\frac{7}{13}x\right)^2 dx$;

pumping distance: $x$

$$W = 9800 \int \left(\begin{matrix}\text{pumping dist.}\\ \text{of slice}\end{matrix}\right) dV = 9800 \int_1^{13} (x)(\pi)\left(\frac{7}{13}\right)^2 x^2 \, dx = 9800\pi\left(\frac{7}{13}\right)^2 \int_1^{13} x^3 \, dx$$

$$= 9800\pi\left(\frac{7}{13}\right)^2 \left[\frac{1}{4}x^4\right]_1^{13} = 9800\pi\left(\frac{7}{13}\right)^2\left(\frac{1}{4}\right)\left[13^4 - 1^4\right] = 63\,735\,814 \text{ J}$$

Method 2. Model the tank using a solid of revolution.




The line passing thru $(7,0)$ and $(0,13)$ is: $y = mx + b$ with $m = \frac{13}{7}$ and $b = 13$

thickness: $dy$, bounds: $y \in [0, 12]$

volume of slice: $dV = \pi r^2 \, dy$ with $r = x$ and $y = \frac{13}{7}x + 13$;

$y - 13 = \frac{13}{7}x$; $x = \frac{7}{13}(y - 13)$;

$dV = \pi\left[\frac{7}{13}(y-13)\right]^2 dy = \pi\left(\frac{7}{13}\right)^2\left(y^2 - 26y + 13^2\right) dy$

pumping distance: $13 - y$

$$W = 9800 \int_0^{12} (13 - y)\pi\left(\frac{7}{13}\right)^2\left(y^2 - 26y + 13^2\right) dy = \cdots = 63\,735\,814 \text{ J}$$